

A Perturbative Approach to Random Matrix Spectra

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Random matrices often find applications in quantum systems whose eigenenergies are not easily solved analytically, so their behavior is better thought of statistically. Such systems may include but are not limited to heavy nuclei [7], quantum wires [4], and quantum dots [2]. A common point of interest in such quantum systems is the set of energy levels which they exhibit, ie. their spectra, because energy level gaps in spectra determine the wavelengths of light that the systems can emit. This paper presents an analysis of the spectra of random Hamiltonian matrices, with applications to the emission spectra of quantum dots, ie. small grains of metal. We combine non-degenerate perturbation theory with ideas about Brownian motion to derive the joint distribution of energy levels, and show that the level spacings can be used to model emission spectra of quantum dot with impurities. In the end, we find that quantum dots have emission spectra which decreases to zero for arbitrarily long wavelengths of light.

I. INTRODUCTION

Most realistic quantum systems have analytically intractable Hamiltonians. For these systems, a natural step to draw insight into the system's behavior is to ignore the Hamiltonian's complexities by treating the Hamiltonian as random, and using statistical estimations of its spectra. Oftentimes, we would like to learn about the emission and absorption spectra of these systems, by studying the energy differences between energy levels of the system, and it is specifically these energy gaps which are modelled well by the energy gaps in random Hamiltonians. Such systems include but are not limited to chaotic billiards [3], metals with irregular impurities [1], quantum dots [2], quantum wires [4], and heavy nuclei [8]. In this paper, we aim to study these spectra by deriving significant results in random matrix theory, with applications towards the emission spectra of quantum dots with impurities.

The rest of the paper is structured as follows. In Section II, we begin from the assumption that the Hamiltonian of the system is represented by a random matrix, and define what a random matrix is. Then in Section III, we will view random Hamiltonians through the lens of perturbation theory to introduce the effect of energy level repulsion. In Section IV, we solve for the distribution of energy level arrangements in random Hamiltonians using the effects of energy level repulsion. In Section V, we show computationally that the spacings between energy levels in a quantum dot are well modelled by the random Hamiltonian. Finally, Section VI discusses potential applications. For readers who are interested, Appendix A includes an additional derivation of a result in random matrix theory known as the Wigner semicircle law.

This paper is presented from a pedagogical view

of random matrix theory, and presumes that readers have some basic probability knowledge. For readers who need to review this, Appendix B states some facts about normal distributions denoted $N(\mu, \sigma^2)$ which are relevant to this paper.

II. RANDOM HAMILTONIAN MATRICES

In order to study the energy gaps between levels in random Hamiltonians, we would first like to specify the exact way in which our Hamiltonians are random. We will consider Hamiltonians over a finite-dimensional state space of the form $H_0 = X^\dagger + X$, where each element of X is drawn independently and identically distributed (iid.) from a complex normal distribution $X_{ij} \sim^{iid} N(0, 1/2) + iN(0, 1/2)$. Under this definition, H_0 has iid elements distributed like $N(0, 2)$ along the diagonal, and iid elements distributed like $N(0, 1) + iN(0, 1)$ on one side of the diagonal. Our assumption that the Hamiltonian comes in this form (or is proportional to this form) is not true for most systems, but it will turn out that despite this, random matrices of this form are still good models to determine the energy gaps between levels in quantum dots. Random Hamiltonians satisfy two useful properties: invariance under unitary conjugation and scaling under random perturbation.

To show invariance under unitary conjugation, take an arbitrary unitary matrix U . Note that the distribution of X is invariant under Hermitian conjugate and multiplication by U . This means that $U^\dagger X U$ also comes from the same distribution as X , implying that $U^\dagger H_0 U = U^\dagger X U + U^\dagger X^\dagger U$ is from the same distribution as H_0 .

To show that H_0 scales under random perturbation, we define a random matrix δH independent of but drawn from the same distribution as H_0 . Then

since the variances in sums of random variables add together, we know that $H_0 + \lambda\delta\tilde{H}$ is distributed the same way as $\sqrt{1 + \lambda^2}H_0$. Therefore, perturbing the Hamiltonian H_0 by the matrix

$$\delta H = \left(\frac{1}{\sqrt{1 + \lambda^2}} - 1 \right) H_0 + \frac{\lambda}{\sqrt{1 + \lambda^2}} \delta\tilde{H} \quad (1)$$

with randomly drawn $\delta\tilde{H}$ leaves the distribution of H_0 unchanged.

III. RANDOMLY PERTURBED ENERGY LEVELS

Our goal is to find the gaps between energy levels in a random Hamiltonian. To do this, we may first solve for the distribution of the spectrum, ie. the set of positions of the energy levels, based on the fact that the random perturbation δH of Equation 1 leaves the distribution of H_0 and consequently the spectrum unchanged. In this section, we start by finding out how the perturbation δH shifts the individual energy levels in the spectrum.

We expand Equation 1 to second order in the small quantity λ to determine the effect of δH on H_0 ,

$$H = H_0 + \delta H = H_0 + \lambda\delta\tilde{H} - \frac{\lambda^2}{2}H_0 + O(\lambda^3) \quad (2)$$

and observe that our perturbation not only has terms linear in λ , but higher order terms as well. Non-degenerate perturbation theory as typically taught in textbooks does not account for higher order terms, so we must first extend the theory to account for these terms. We write out the eigenstates and energy levels as power series in the perturbation strength λ ,

$$|\psi_n\rangle = |\psi_n^{(0)}\rangle + \lambda |\psi_n^{(1)}\rangle + \lambda^2 |\psi_n^{(2)}\rangle + O(\lambda^3) \quad (3)$$

$$E_n = E_n^{(0)} + \lambda E_n^{(1)} + \lambda^2 E_n^{(2)} + O(\lambda^3) \quad (4)$$

and expand the eigenequation,

$$H |\psi_n\rangle = E_n |\psi_n\rangle \quad (5)$$

$$\begin{aligned} & \left(H_0 + \lambda\delta\tilde{H} - \frac{\lambda^2}{2}H_0 + O(\lambda^3) \right) \\ & \left(|\psi_n^{(0)}\rangle + \lambda |\psi_n^{(1)}\rangle + \lambda^2 |\psi_n^{(2)}\rangle + O(\lambda^3) \right) \\ & = \left(E_n^{(0)} + \lambda E_n^{(1)} + \lambda^2 E_n^{(2)} + O(\lambda^3) \right) \\ & \left(|\psi_n^{(0)}\rangle + \lambda |\psi_n^{(1)}\rangle + \lambda^2 |\psi_n^{(2)}\rangle + O(\lambda^3) \right) \end{aligned} \quad (6)$$

collecting matching powers of λ and deriving the first and second order energy corrections:

$$E_n^{(1)} = \delta\tilde{H}_{nn} \quad (7)$$

$$|\psi_n^{(1)}\rangle = \sum_{k \neq n} \frac{\delta\tilde{H}_{kn}}{E_n^{(0)} - E_k^{(0)}} |k^{(0)}\rangle \quad (8)$$

$$E_n^{(2)} = -\frac{E_n^{(0)}}{2} + \sum_{k \neq n} \frac{|\delta\tilde{H}_{kn}|^2}{E_n^{(0)} - E_k^{(0)}} \quad (9)$$

The total perturbative effect produces energy level shifts of

$$\begin{aligned} E_n &= E_n^{(0)} + \lambda\delta\tilde{H}_{nn} \\ &+ \lambda^2 \left(-\frac{E_n^{(0)}}{2} + \sum_{k \neq n} \frac{|\delta\tilde{H}_{nk}|^2}{E_n^{(0)} - E_k^{(0)}} \right) \\ &+ O(\lambda^3). \end{aligned} \quad (10)$$

which is determined by the random $\delta\tilde{H}$. These energy level shifts depend on the matrix elements of $\delta\tilde{H}$ in the H_0 eigenbasis. Now, note that since H_0 retains its distribution under unitary conjugation but $\delta\tilde{H}$ has the same distribution as H_0 , $\delta\tilde{H}$ in the eigenbasis of H_0 has the same form as the original distribution. In other words, the diagonal still consists of $\delta\tilde{H}_{nn} \sim \text{iid } N(0, 2)$ values and one side of the diagonal still consists of $\delta\tilde{H}_{nk} \sim \text{iid } N(0, 1) + iN(0, 1)$.

We may interpret total perturbative effect of Equation 10 as follows:

- $\lambda\delta\tilde{H}_{nn}$ causes energy level n to move randomly.
- $-\lambda^2 E_n^{(0)}/2$ causes energy level n to drift towards zero.
- $\lambda^2 \sum_{k \neq n} |\delta\tilde{H}_{nk}|^2 / (E_n^{(0)} - E_k^{(0)})$ causes energy levels n and k to repel and drift away from each other, with an ‘‘inverse distance’’ law.

All of these effects are independent, except for the energy level repulsion, where pairwise repulsions have an equal effect on both energy levels. The energy level repulsion explodes only when the energy levels are close to one another and the drift towards zero explodes only when the energy levels are far from zero. Therefore, the energy levels must converge to some confined joint distribution which is neither arbitrarily concentrated nor spread out. This is expected because the original random Hamiltonian H_0 which exhibits these levels is of the scale $O(1)$.

IV. FIXED JOINT DISTRIBUTION OF ENERGY LEVELS

In this section, we figure out how the random shifting of the spectrum in the random Hamiltonian leads the spectrum to a fixed distribution. Further random shifting of the spectrum no longer changes its distribution.

For ease of study, we will write the spectrum as the vector of energy eigenvalues \mathbf{E} with components E_n , which collects the energy levels together into a single vector. Using the independence of matrix elements of $\delta\bar{H}$, we find from Equation 10 that the mean and covariance in perturbations of \mathbf{E} is given by

$$\mathbb{E}(E_n - E_n^{(0)}) = \lambda^2 \mu_n(\mathbf{E}^{(0)}) + O(\lambda^3) \quad (11)$$

$$\mu_n(\mathbf{E}^{(0)}) = -\frac{E_n^{(0)}}{2} + \sum_{k \neq n} \frac{2}{E_n^{(0)} - E_k^{(0)}} \quad (12)$$

$$\mathbb{E}((E_n - E_n^{(0)})(E_m - E_m^{(0)})) = O(\lambda^4) \quad (13)$$

$$\mathbb{E}((E_n^2 - E_n^{(0)2})) = 2\lambda^2 + O(\lambda^4) \quad (14)$$

We write $\mu_n(\mathbf{E}^{(0)})$ to abbreviate the coefficient of λ^2 in Equation (11) as a function of the vector of energy levels, and let $\boldsymbol{\mu}$ be the vector (μ_1, \dots, μ_N) . Note that the perturbation induces both drift and variance on the spectrum \mathbf{E} , of order λ^2 . Furthermore, there is no correlation between energy level shifts in $\Delta\mathbf{E} = \mathbf{E} - \mathbf{E}^{(0)}$ to second order. Notice that according to Equation 10, $\Delta\mathbf{E}$ is normally distributed for small λ . If one applies n successive independent perturbations of strength $\lambda^2 = 1/n$, then for large n , then the motion of the spectrum \mathbf{E} behaves much like Brownian motion with drift, but where the drift is a function of \mathbf{E} . [9] Therefore, we will perform any further calculations with small λ by approximating the distribution of $\Delta\mathbf{E}$ using a multivariate normal distribution of mean $\boldsymbol{\mu}(\mathbf{E})$ and variance $2\lambda\mathbf{I}$.

Now that we have characterized the distribution of shifts $\Delta\mathbf{E}$, we would like to find the distribution $P(\mathbf{E})$ of spectra which is fixed under this random shift. Under this fixed distribution, we must have equal probability of shifting to and from any given spectrum \mathbf{E} . In other words, we have

$$\begin{aligned} & \int P(\mathbf{E}|\mathbf{E}_{\text{other}})P(\mathbf{E}_{\text{other}})d\mathbf{E}_{\text{other}} \\ &= \int P(\mathbf{E}_{\text{other}}|\mathbf{E})P(\mathbf{E})d\mathbf{E}_{\text{other}} \end{aligned} \quad (15)$$

where $P(\mathbf{E}_1|\mathbf{E}_2)$ denotes the probability density of shifting to spectrum \mathbf{E}_1 given that the spectrum began at \mathbf{E}_2 . To find the fixed distribution, we will

make an approximating assumption which takes inspiration from the Metropolis algorithm [5] from computer science. This based on the idea that if the shift probabilities are set up such that we can satisfy the Bayes rule,

$$\frac{P(\mathbf{E}_{\text{final}}|\mathbf{E}_{\text{initial}})}{P(\mathbf{E}_{\text{initial}}|\mathbf{E}_{\text{final}})} = \frac{P(\mathbf{E}_{\text{final}})}{P(\mathbf{E}_{\text{initial}})} \quad (16)$$

for some $P(\mathbf{E})$, then this would satisfy Equation 15 implying $P(\mathbf{E})$ is the desired fixed distribution of spectra. Therefore, we can use the known shift probabilities on the left side of Equation 16 to deduce information about the spectrum distribution $P(\mathbf{E})$ on the right. We do this by figuring out how small perturbations cause the probability density of spectra to flow in \mathbf{E} space, to first order in λ . To characterize the left side, define $\mathbf{E}(\epsilon) = \mathbf{E}^{(0)} + \epsilon\Delta\mathbf{E}$ for some spectrum $\mathbf{E}_{\text{initial}} = \mathbf{E}^{(0)}$ and shift $\Delta\mathbf{E}$. Due to our approximation of $\Delta\mathbf{E}$ as normally distributed, the ratio of probabilities of transitioning from $\mathbf{E}_{\text{initial}} = \mathbf{E}(0)$ to $\mathbf{E}_{\text{final}} = \mathbf{E}(\lambda)$ and the reverse behaves for small λ according to

$$\lim_{\lambda \rightarrow 0} \frac{\ln \frac{P(\mathbf{E}(\lambda)|\mathbf{E}(0))}{P(\mathbf{E}(0)|\mathbf{E}(\lambda))}}{\lambda} \quad (17)$$

$$= \lim_{\lambda \rightarrow 0} \frac{\ln \frac{\exp(-\frac{1}{2\lambda^2}(\lambda\Delta\mathbf{E} - \lambda^2\boldsymbol{\mu}(\mathbf{E}(0)))^T(2\mathbf{I})^{-1}(\lambda\Delta\mathbf{E} - \lambda^2\boldsymbol{\mu}(\mathbf{E}(0))))}{\exp(-\frac{1}{2\lambda^2}(-\lambda\Delta\mathbf{E} - \lambda^2\boldsymbol{\mu}(\mathbf{E}(\lambda)))^T(2\mathbf{I})^{-1}(-\lambda\Delta\mathbf{E} - \lambda^2\boldsymbol{\mu}(\mathbf{E}(\lambda))))}}{\lambda}} \quad (18)$$

$$\begin{aligned} &= \lim_{\lambda \rightarrow 0} \frac{1}{4\lambda} (\Delta\mathbf{E} + \lambda\boldsymbol{\mu}(\mathbf{E}(\lambda))) \cdot (\Delta\mathbf{E} + \lambda\boldsymbol{\mu}(\mathbf{E}(\lambda))) \\ &\quad - \frac{1}{4\lambda} (\Delta\mathbf{E} - \lambda\boldsymbol{\mu}(\mathbf{E}(0))) \cdot (\Delta\mathbf{E} - \lambda\boldsymbol{\mu}(\mathbf{E}(0))) \end{aligned} \quad (19)$$

$$= \boldsymbol{\mu}(\mathbf{E}(0)) \cdot \Delta\mathbf{E} \quad (20)$$

This quantifies how much more likely the spectrum is to shift towards the drift versus against it. Because of this, the fixed distribution $P(\mathbf{E})$ satisfies

$$\left(\frac{d}{d\mathbf{E}} \ln P(\mathbf{E}) \right) \cdot \Delta\mathbf{E} \quad (21)$$

$$= \lim_{\lambda \rightarrow 0} \frac{\ln P(\mathbf{E}^{(0)} + \lambda\Delta\mathbf{E}) - \ln P(\mathbf{E}^{(0)})}{\lambda} \quad (22)$$

$$= \lim_{\lambda \rightarrow 0} \frac{\ln \frac{P(\mathbf{E}(\lambda))}{P(\mathbf{E}(0))}}{\lambda} \quad (23)$$

$$= \lim_{\lambda \rightarrow 0} \frac{\ln \frac{P(\mathbf{E}(\lambda)|\mathbf{E}(0))}{P(\mathbf{E}(0)|\mathbf{E}(\lambda))}}{\lambda} \quad (24)$$

$$= \boldsymbol{\mu}(\mathbf{E}) \cdot \Delta\mathbf{E} \quad (25)$$

$$\frac{d}{d\mathbf{E}} \ln P(\mathbf{E}) = \boldsymbol{\mu}(\mathbf{E}). \quad (26)$$

or in other words, the distribution of spectra bunches up more in the direction of the spectrum's drift. Then using the drift direction and strength, we can solve for $P(\mathbf{E})$. Note that since $\frac{\partial \mu_n}{\partial E_n^{(0)}} - \frac{\partial \mu_m}{\partial E_n^{(0)}} = 0$, it is possible to write $\boldsymbol{\mu}$ as the gradient of a scalar function of \mathbf{E} . According to Equation 26, this function is the log of $P(\mathbf{E})$. This finally gives us $P(\mathbf{E})$, ie. the probability density of the spectrum of a random Hamiltonian H_0 ,

$$P(\mathbf{E}) = C_N \prod_n e^{-\frac{(E_n^{(0)})^2}{4}} \prod_{k < n} |E_n^{(0)} - E_k^{(0)}| \quad (27)$$

for some normalization C_N dependent on the number of energy levels N . This normalization can be computed using Mehta's integral [6]. The mutual repulsion of energy levels manifests itself in the fact that the probability density is higher when no pair of energy levels are close to each other. In fact, there is a probability density of zero that two energy levels are equal.

Now that we have determined the distribution of energy level spectra for random Hamiltonians, we may study applications of this distribution to quantum dots.

V. ENERGY LEVEL GAPS IN QUANTUM DOTS

Physical systems such as quantum dots, ie. small grains of metal, are capable of emitting energy in the form of photons. UV light can be used to excite electrons in the dots, and with every electron de-excitation towards the ground state a photon is released with energy equal to the energy gap between the initial and final state of the de-excitation. This means that the energy differences between various levels is what determines the emission spectra of the quantum dots. More specifically, levels which are closer together will emit longer wavelengths of light upon transitioning. Quantum dots are the subject of ongoing research, some of which aims to use them as laser mediums and efficient light sources. By studying the gaps between levels in quantum dots, we may then learn about the emission spectra of quantum dots. In this section, we show that random matrices provide a good model of these energy gaps.

Our model for a quantum dot is a two-dimensional infinite well in the shape of a metallic grain, which may be irregular. The wavefunction of the electron is thus confined to the interior of this grain, and the Hamiltonian is

$$H = \frac{p_x^2 + p_y^2}{2m} + V(x, y) \quad (28)$$

where $V(x, y)$ is 0 within the grain and ∞ everywhere else. This model is known as a quantum billiard, because in the classical limit, it models a particle which bounces off the wall like a billiard ball. It is hypothesized that if the classical billiard exhibits a phenomenon known as "chaos", the quantum system will have level gaps distributed in the same way as in a random matrix. [3] In 1957, Wigner estimated that at energy E , the probability distribution $p(\Delta E)$ of gaps ΔE between consecutive levels in random Hamiltonians is given by

$$p(\Delta E) = \frac{\pi \rho(E)^2 \Delta E}{2} e^{-\frac{\pi \rho(E)^2 \Delta E^2}{4}} \quad (29)$$

where $\rho(E)$ is the average level density at energy E . Our aim is to show that this distribution approximately holds true for quantum dots as well, if they are shaped like chaotic billiards. The level density $\rho(E)$ for random Hamiltonians is given by the Wigner semicircle law [7], which is derived in Appendix A. The true distribution of level spacings is difficult but possible to derive from Equation 27 and is not exactly equal to Wigner's estimate in Equation 29, but the estimate is close enough for our study nonetheless.

For the purposes of this analysis, we analyze a grain which is square of length L in shape, but with 20 randomly placed atomic defects modelled as small infinite circular barriers covering 0.01% of the square's area each, as shown in Figure 2. The presence of these barriers causes the grain to become a chaotic quantum billiard. If these barriers are not included, then the grain's level density $\rho(E)$ can be solved analytically and turns out to be uniform:

$$\psi_{n_1, n_2}(x, y) \propto \sin(\pi x/L) \sin(\pi y/L) \quad (30)$$

$$E_{n_1, n_2} = \hbar^2 \pi^2 (n_1^2 + n_2^2) / 2mL^2 \quad (31)$$

$$\rho(E) = \int_0^\infty \int_0^\infty \delta(E_{n_1, n_2} - E) dn_1 dn_2 \quad (32)$$

$$= \frac{mL^2}{2\pi\hbar^2} \quad (33)$$

For the grain with defects, we may assume that the level density $\rho(E)$ is approximately the same, and perform a computational simulation to determine the distribution $p(\Delta E)$ of level spacings. By discretizing space to a 100×100 grid to numerically approximate the system with the defects included, we approximate the true Hamiltonian with a $100 \times 100 \times (1 - 0.01\% \times 20) = 9980$ dimensional Hamiltonian, and eigendecompose it to compute all the energy levels of the grain and then the gaps between consecutive levels. We discard gaps formed by levels in the lowest quarter of energies to avoid

effects caused by the edge of the energy distribution, and the upper half of energies to avoid inaccuracies caused by space discretization in the simulation. The distribution of the remaining quarter of gaps, shown in Figure 1, verify that the energy gaps between consecutive levels indeed follow the same distribution as energy gaps in random Hamiltonians. This verifies that random Hamiltonians serve as good models when studying the emission spectra of impure quantum dots. Since in random Hamiltonians, the probability of finding an energy gap of small size decreases proportionally to the size, we may deduce that the same is true for the quantum dot, and thus impure quantum dots tend to emit exceedingly small amounts of light at long wavelengths.

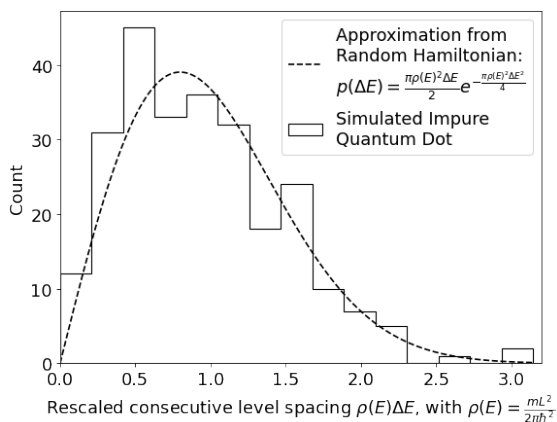


Figure 1: This histogram shows the calculated distribution of N energy gaps ΔE in a 0.2% impure quantum dot depicted in Figure 2. Overlaid is Wigner’s predicted distribution of energy gaps in random Hamiltonians, from Equation 29. The approximate agreement between the two indicates that random Hamiltonians are suitable for modelling the emission spectra of these quantum dots.

VI. DISCUSSION

The analysis we presented in this paper uses the physics of non-degenerate perturbation theory to introduce major results in random matrix theory such as the joint distribution of energy levels in Equation 27, the Wigner semicircle law in Equation A13, and the energy gap distribution in Equation 29. Our nontraditional approach to random matrices illustrated how Brownian motion of the Hamiltonian allows the distribution of spectra to smooth and flow towards a fixed distribution over time. The repulsion of energy levels during this flow leads to a ten-

dency for the levels to space themselves apart, such that there is a probability density of zero that the random matrix has degeneracies.



Figure 2: The quantum dot used in our simulation is a small metal grain with 20 randomly placed impurities, each covering 0.01% of the area of the grain. An electron travels freely within this grain, which is modelled by a potential $V(x, y)$ of ∞ within the impurities or outside the grain, and 0 elsewhere. The distribution of spacings between consecutive energy levels of the electron is shown in Figure 1.

We have also shown that random Hamiltonians serve as a good model for the behavior of energy level gaps in quantum dots, which have potential uses as laser media and efficient light sources in quantum light emitting diodes (QLEDs). Quantum dots are modelled as chaotic billiards, which are hypothesized to have the same level spacing distributions as random Hamiltonians, and this is empirically true. Since there are fewer spacings as the spacing size grows small, quantum dots tend to emit less light at very long wavelengths than at moderate wavelengths.

In conclusion, the perturbation theory can be used to study Hamiltonians represented by random matrices, yielding random matrix theoretical results applicable to the physics of quantum dots.

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- [9] In the limit as $n \rightarrow \infty$, the motion of \mathbf{E} is an Itô process described by a stochastic differential equation. The probability density of \mathbf{E} then evolves according to the Fokker-Planck equation, whose fixed point can be solved to shortcut to Equation 27. We do not attempt this derivation because it is difficult without sufficient background in probability, though our analysis follows steps which mirror a derivation of the Fokker-Planck equation for our simplified case.

Appendix A: Wigner Semicircle Law

Many physical systems have a huge, potentially infinite state space, so it makes sense to study random Hamiltonians of large dimension. In this section having already determined how the spectrum is distributed in Equation 27, we will find that the energy level density for large random Hamiltonians follows the Wigner semicircle distribution.[7] The energy level density can be used as a normalization when studying the statistics of the energy level gaps like in Section V.

We take the limit as the number of states N goes to infinity, and assume that the energy level density at energy $\sqrt{N}E$ goes like $\sqrt{N}f(E) + o(\sqrt{N})$ for some desired fixed function f which integrates to 1. This assumption will later turn out to be justified when we indeed find such a function f which is well-behaved. Our goal is then to try to find this

function f . Without loss of generality, let f be zero outside of interval $(-E_0, E_0)$ for some E_0 which is not necessarily finite. Note that f is symmetrical because we are equally likely to sample a Hamiltonian H versus $-H$, which have the negative of each others’ energy levels, and thus reversed energy level densities. We can then write the probability $P(f)$ that the energy level density function is f by converting sums to integrals:

$$\ln P(\mathbf{E}) = C_N - \frac{1}{4} \sum_n E_n^2 + \sum_{k \leq n} \ln |E_n - E_k| \quad (\text{A1})$$

$$\begin{aligned} \ln P(f) = & C_N \\ & - \frac{1}{4} \int_{-\sqrt{N}E_0}^{\sqrt{N}E_0} dE E^2 \sqrt{N} f \left(\frac{E}{\sqrt{N}} \right) \\ & + \frac{1}{2} \int_{-\sqrt{N}E_0}^{\sqrt{N}E_0} \int_{-\sqrt{N}E_0}^{\sqrt{N}E_0} dE_1 dE_2 \\ & N f \left(\frac{E_1}{\sqrt{N}} \right) f \left(\frac{E_2}{\sqrt{N}} \right) \ln |E_1 - E_2| \quad (\text{A2}) \end{aligned}$$

$$= C_N + \frac{N^2}{2} \ln N + N^2 g(f) \quad (\text{A3})$$

$$\begin{aligned} g(f) = & \frac{1}{2} \int_{-E_0}^{E_0} \int_{-E_0}^{E_0} dE_1 dE_2 \\ & f(E_1) f(E_2) \ln |E_1 - E_2| \\ & - \frac{1}{4} \int_{-E_0}^{E_0} E^2 f(E) dE \quad (\text{A4}) \end{aligned}$$

Here, the scalar value indicated by $-g(f)$ acts like a “repulsive potential” which regulates the shape of the density function f . Now suppose that the most probable energy level density function is given by some function f^* . Then $g(f^*) > g(f)$ because g is strictly concave on the space of normalized f . This ensures that f^* is infinitely more probable than any other possible density function f ,

$$\lim_{N \rightarrow \infty} \frac{P(f^*)}{P(f)} = \lim_{N \rightarrow \infty} \exp(\ln P(f^*) - \ln P(f)) \quad (\text{A5})$$

$$\begin{aligned} & = \lim_{N \rightarrow \infty} \exp(N^2(g(f^*) - g(f))) \quad (\text{A6}) \\ & \rightarrow \infty. \quad (\text{A7}) \end{aligned}$$

which suggests that the most likely density law maximizes the potential $g(f^*)$. We may then solve for f^* using the calculus of variations, or in other words, by imposing the condition that no change $\epsilon \Delta f$ to the density function f^* within the interval $(-E_0, E_0)$

can improve the value of the potential $g(f^* + \epsilon\Delta f)$,

$$\forall \Delta f : 0 = \frac{d}{d\epsilon} g(f^* + \epsilon\Delta f) \quad (\text{A8})$$

$$\begin{aligned} \frac{d}{d\epsilon} \int_{-E_0}^{E_0} E_1^2 (f^*(E_1) + \epsilon\Delta f(E_1)) dE_1 \\ = 2 \frac{d}{d\epsilon} \int_{-E_0}^{E_0} \int_{-E_0}^{E_0} dE_1 dE_2 \\ (f^*(E_1) + \epsilon\Delta f(E_1)) \\ (f^*(E_2) + \epsilon\Delta f(E_2)) \ln |E_1 - E_2| \quad (\text{A9}) \end{aligned}$$

$$\begin{aligned} \int_{-E_0}^{E_0} dE_1 \Delta f(E_1) \\ \left(E_1^2 - 4 \int_{-E_0}^{E_0} f^*(E_2) \ln |E_1 - E_2| dE_2 \right) \\ = 0 \quad (\text{A10}) \end{aligned}$$

Since this above equation must hold for all Δf which integrate to 0 over $(-E_0, E_0)$, we must have the equation

$$4 \int_{-E_0}^{E_0} f^*(E_2) \ln |E_1 - E_2| dE_2 = E_1^2 + C \quad (\text{A11})$$

for all $E_1 \in [-E_0, E_0]$ and with a constant C , and this is solved when the energy level density f^* follows a semicircular shape with finite E_0 ,

$$f^*(E) = \frac{1}{2\pi} \sqrt{E_0^2 - E^2} \quad (\text{A12})$$

$$= \frac{1}{2\pi} \sqrt{4 - y^2}, \quad E_0 = 2 \quad (\text{A13})$$

and E_0 is determined to be 2 by normalization of f^* . This is called the Wigner semicircle law, and was first found by Wigner in [7]. This law is more mathematically significant and less physically significant, since most physical systems exhibit a different level density function and the focus is on the emission spectra derived from statistics of individual energy level differences, rather than the global distribution of energy levels.

Appendix B: Properties of Random Variables and Normal Distributions

This section is a list of basic facts about probability and normal distributions which are relevant to this paper.

- Expectation is linear. If we have random variables X and Y , then $\mathbb{E}(X+Y) = \mathbb{E}(X) + \mathbb{E}(Y)$.

- If two random variables are independent, their joint probability distributions factorize into the product of their individual probability distributions.

- The acronym iid. stands for ‘‘independent and identically distributed’’.

- The covariance of a random vector \mathbf{x} is the matrix $\mathbb{E}((x - \mathbb{E}(x))(x - \mathbb{E}(x))^\dagger)$, and the variance σ^2 of a random scalar is $\mathbb{E}(|x|^2)$. Since expectations are linear, covariances and variances add when their respective random variables add and are all mutually independent.

- The normal distribution denoted $N(\mu, \sigma^2)$ with mean μ and variance σ^2 is a probability distribution over the real numbers with probability density function

$$p(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2}.$$

- The complex normal distribution with mean μ and variance σ^2 is a probability distribution over the complex numbers with probability density function

$$p(z) = \frac{1}{\pi\sigma^2} e^{-|z-\mu|^2/\sigma^2}.$$

The unit complex normal distribution is a complex normal distribution of mean $\mu = 0$ and variance $\sigma^2 = 1$ and is written in this paper as $N(0, 1/2) + iN(0, 1/2)$.

- When adding real or complex normally distributed variables, the resulting sum’s mean and (co)variance are the sum of each variables’ means and (co)variances.

- Multiplying a random variable by a constant x causes its mean to scale by x and its variance to scale by x^2 .

- Adding two normally distributed random variables or rescaling one normally distributed random variable results in another normally distributed random variable.

- A random vector of iid. real (resp. complex) normally distributed values has a spherically symmetric distribution, which is therefore unaffected when multiplied by an orthogonal (resp. unitary) matrix.

- The multivariate real normal distribution of N variables, mean $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$ has a probability density function of

$$\frac{\exp(-\frac{1}{2}(\mathbf{x} - \boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1}(\mathbf{x} - \boldsymbol{\mu}))}{\sqrt{\det(2\pi\boldsymbol{\Sigma})}} \quad (\text{B1})$$